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ABSTRACT

Excerpts from interviews with pupils in grades 1, 2, 3, and 6 to ascertain their interpretations of mathematical sentences and symbols are presented. Children were found to consider the equal sign as an operator symbol ("do something") and not as a relational symbol. Implications for teaching are briefly discussed. (MS)

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No. 3

How Children View Equality Sentences

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pmdc

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FOREWORD

Ed Begle recently remarked that curricular efforts during the 1960's taught us a great deal about how to teach better mathematics, but very little about how to teach mathematics better. The mathematician will, quite likely, agree with both parts of this statement. The layman, the parent, and the elementary school teacher, however, question the thesis that the "new math" was really better than the "old math." At best, the fruits of the mathematics curriculum "revolution" were not sweet. Many judge them to be bitter.

While some viewed the curricular changes of the 1960's to be "revolutionary," others disagreed. Thomas C. O'Brien of Southern Illinois University at Edwardsville recently wrote, "We have not made any fundamental change in school mathematics."[1] He cites Allendoerfer who suggested that a curriculum which heeds the ways in which young children learn mathematics is needed. Such a curriculum would be based on the understanding of children's thinking and learning. It is one thing, however, to recognize that a conceptual model for mathematics curriculum is sound and necessary and to ask that the child's thinking and learning processes be heeded, it is quite another to translate these ideas into a curriculum which can be used effectively by the ordinary elementary school teacher working in the ordinary elementary school classroom.

Moreover, to propose that children's thinking processes should serve as a basis for curriculum development is to presuppose that curriculum makers agree on what these processes are. This is not the case, but even if it were, curriculum makers do not agree on the implications which the understanding of these thinking processes would have for curriculum development.

In the real world of today's elementary school classroom, where not much hope for drastic changes for the better can be foreseen, it appears that in order to build a realistic, yet sound basis for the mathematics curriculum, children's mathematical thinking must be studied intensively in their usual school habitat. Given an opportunity to think freely, children clearly display certain patterns of thought as they deal with ordinary mathematical situations encountered daily in their classroom. A videotaped record of the outward manifestations of a child's thinking, uninfluenced by any teaching on the part of the interviewer, provides a rich source for conjectures as to what this thinking is, what mental structures the child has developed, and how the child uses these structures when dealing with the ordinary concepts of arithmetic. In addition, an intensive analysis of this videotape generates some conjectures as to the possible sources of what adults view as children's "misconceptions" and about how the school environment (the teacher and the materials) "fights" the child's natural thought processes.

The Project for the Mathematical Development of Children (PMDC)[2] set out to create a more extensive and reliable basis on which to build mathematics curriculum. Accordingly, the emphasis in the first phase is to try to understand the children's intellectual pursuits, specifically their attempts to acquire some basic mathematical skills and concepts.

The PMDC, in its initial phase, works with children in grades 1 and 2. These grades seem to comprise the crucial years for the development of bases for the future learning of mathematics, since key mathematical concepts begin to form at these grade levels. The children's mathematical development is studied by means of:

1. One-to-one videotaped interviews subsequently analyzed by various individuals.

2. Teaching experiments in which specific variables are observed in a group teaching setting with five to fourteen children.

3. Intensive observations of children in their regular classroom setting.

4. Studies designed to investigate intensively the effect of a particular variable or medium on communicating mathematics to young children.

1 "Why Teach Mathematics?" The Elementary School Journal 73 (Feb. 1973), 258-68.

2 PMDC is supported by the National Science Foundation, Grant No. PES 74-18106-A03.

5. Formal testing, both group and one-to-one, designed to provide further insights into young children's mathematical knowledge.

The PMDC staff and the Advisory Board wish to report the Project's activities and findings to all who are interested in mathematical education. One means for accomplishing this is the PMDC publication program.

Many individuals contributed to the activities of PMDC. Its Advisory Board members are: Edward Begle, Edgar Edwards, Walter Dick, Renee Henry, John LeBlanc, Gerald Rising, Charles Smock, Stephen Willoughby and Lauren Woodby. The principal investigators are: Merlyn Behr, Tom Denmark, Stanley Erlwanger, Janice Flake, Larry Hatfield, William McKillip, Eugene D. Nichols, Leonard Pikaart, Leslie Steffe, and the Evaluator, Ray Carry. A special recognition for this publication is given to the PMDC Publications Committee, consisting of Merlyn Behr (Chairman), Thomas Cooney and Tom Denmark.

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Eugene D. Nichols,
Director of PMDC

In the early elementary school grades, much of the number work is presented to children in the form of open number sentences. In learning the basic addition facts, for example, children are expected to respond to such sentences as $3 + 4 = \square$. The question of what meaning, if any, symbolic sentences such as these have to children is of concern in the research discussed in this article. In order for teachers to communicate effectively with children and teach in a manner that enhances understanding, they need to know what meaning such sentences have to children.

The mathematical symbols which children encounter early in the learning of arithmetic are +, =, and such expressions as 3 + 4. The symbol = has several meanings to adults. The most basic is probably that it is an abstraction of the notion of *sameness*. This is an intuitive notion of equality which arises from experience with sameness of the numerousness of sets of objects in the real world. This is the notion of equality which we would hope children would exhibit. A more sophisticated notion of equality, which comes as a result of teaching, is that it is an *equivalence relation*. This means that such statements as $1 = 1$, $2 = 2$, $3 = 3$, etc. are true in all cases. Similarly, using any whole numbers, such statements as *if* $2 + 1 = 3$, *then* $3 = 2 + 1$, are true statements. And finally, such statements as *if* $3 = 2 + 1$ *and* $2 + 1 = 1 + 2$, *then* $3 = 1 + 2$ are also true for all whole numbers. These are instances of reflexive, symmetric, and transitive properties, respectively.

In conducting research of concern in this paper, we raised questions like the following:

(1) Must addition sentences be of a certain form in order to be considered true by children?

(2) Are the sentences $2 + 4 = \square$ and $\square = 2 + 4$ viewed as having the same meaning?

We also investigated children's understanding of such sentences as $3 = 5$, $3 = 3$, and $2 + 3 = 3 + 2$.

Some insight into children's ideas about such sentences was gained through non-structured individual interviews with first through sixth graders. We present a number of episodes from some of the interviews, summaries of others, and some discussion of children's remarks. In the reporting, remarks enclosed inside brackets describe behaviors on the part of the interviewer or the child. A series of six dots (. . . . . .) in a transcript indicates that some statements have been omitted; the bracketed [and] in this same context indicates that the statements of the child are joined and related but a question or comment from the interviewer may have intervened.

FIRST AND SECOND GRADERS' IDEAS ABOUT EQUALITY AND ADDITION

We present several first and second graders' reactions to questions about the symbols in written addition sentences of the form $a + b = \square$, where a and b are small whole numbers. We begin with an interview episode with a first grader K (IQ 139[1]) about $2 + 4 = \square$. B is the interviewer.

B: You read that [+], will you? What does that sign say?

K: Plus.

B: What does it tell you to do?

K: It tells you to add; it tells you to do this [touches 2] and this [touches 4].

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[1] IQ reported is from Stanford-Binet short form.

B: O. K. What can you tell me about that [=] symbol?

K: Equal . . . [and] . . . that means, like *this 2 plus . . . plus this 4 equals 6.* There has to be an equal right there.

Two other first graders, who were presented with sentences of the same form, responded in the following manner. C (IQ 108), when presented with 1 + 3 = □, says "I know what it is," and writes 4 in □. He reads: "1 plus 3 equals 4." But, when questioned about the meaning of + and 1 + 3 he says, "It's a hardy. I don't know." He is unable to say what = means. E (IQ 111) when presented with 2 + 4 = □ says that it means to put number 6 in there [□]. E accepts 2 and 4 as numbers but not 2 + 4 "because you put a plus there." When she is asked about the meaning of +, =, and 2 + 4, she says she does not know.

Two second graders react as follows: M (IQ 136) is able to judge such sentences as 2 + 3 = 5 and 2 + 3 = 7 as true or false. He accepts 2 and 4 as being numbers and says that 2 + 4 is a number because if "you put them both together, it makes another number." He explains that the + means "adding" and attempts to describe what = means by saying: "when two numbers are added, that's what it [answer] turns out to be." D (IQ 128) is able to judge sentences like 2 + 3 = 5 and 2 + 3 = 7 as true or false. She says that 2 and 4 are numbers but about 2 + 4 she says, "I don't know, the + means you add together to a number a number"; the = means "what . . . the numbers equal up to." Thus, in 3 + 4 = 7, "the = sign means that, what it adds up to."

All of the children interviewed were able to solve addition sentences. Furthermore, K, a bright first grader, and the second graders accept expressions like 2 + 4 as meaningful, but this configuration of symbols suggests that something must be done. They do not, for example, think of 2 + 4 as being a name for six. The first graders C and E have a less mature understanding of +, =, and 2 + 4. They are unable to tell what these symbols mean when used either individually or in the context of a mathematical sentence.

The above observations suggest that when children see a statement like 3 + 4 = □, they perceive it as a stimulus calling for an answer to be placed in the box.

HOW CHILDREN REACT TO SENTENCES OF THE FORM □ = a + b

K's view of the sentences of the form □ = a + b is typical of almost all first and second graders who were interviewed. It is expressed in the following brief excerpt from one interview.

B: How do you think you would read that [□ = 3 + 4]?

K: . . . (pause) . . . Blank . . . blank equals 3 plus 4.

B: O. K. What can you say about that, anything?

K: It's backwards! [changes □ = 3 + 4 to 4 + 3 = □] . . . .

B: Suppose I write [writes □ = 2 + 5].

K: Now that plus [touches +] has to be right there [touching =]; and that equals [touching =] has to be right there [touching +]; and I'm trying to add up to 5 [K changes □ = 2 + 5 to □ + 2 = 5].

We observe that K reacts to sentences like □ = 4 + 5 in two ways: She says it's backwards and rewrites it as 5 + 4 = □ or interchanges the + and = so it becomes □ + 4 = 5. Later in the interview she comments about □ = 3 + 4 saying, "See, you asked the question backwards . . . you can't go 7 equals 3 plus 4." Thus, although K can read and solve some of these types of sentences, she has definite ideas about how they should be written.

Let's look at how some other first graders react to sentences of this form. When C is presented with $\square = 1 + 2$, he writes $3 = 1 + 2$ and reads this as "2 plus 1 equals 3." Given $\square = 3 + 5$, E counts on her fingers, writes $8 = 3 + 5$ and reads "5 plus 3 equals 8." Given $6 = 4 + 1$, she changes it to $6 = 4 + 10$ saying "6 and 4 makes 10." But when she is asked to read $6 = 4 + 10$, she says, "I wrote it wrong" and changes the sentence to $5 = 4 + 1$ and reads "1 plus 4 equals 5." Again, when given $3 = 2 + 1$, she says "You should put a 5 here [i.e., at 1] . . . 'cause that doesn't equal, that doesn't make!" But when asked to read it, she says, "Three . . . (pause) . . . yeah, that's . . . I was reading the wrong way again." E explains that $2 + 3 = 5$ is easier than $5 = 2 + 3$ "because it's [5 =] on this side, and I'm used to having it on that side [= 5]."

Another first grader, T (IQ 126) reacts to ___ = 2 + 5 by scribbling over ___ =, and he changes ___ = 2 + 5 to 2 + 5 = ___. He explains that ___ = 2 + 5 is "backwards" and asks the interviewer, "Do you read backwards?"

The second graders M and D react as follows: When M is given $\square = 3 + 5$, he writes $8 = 3 + 5$ and reads "8 equals 3 plus 5." On the other hand, D also writes $8 = 3 + 5$, but reads it "5 plus 3 equals 8." He shows consistency in that, presented with $5 = 2 + 3$, he reads it "2 plus 3 equals 5." But, given $3 + 5 = 8$, he reads it "3 plus 5 equals 8."

We observe that only the second grader M accepted a sentence like $\square = 2 + 5$. The other children resist sentences of this form and change them to the forms $2 + 5 = \square$ or $\square + 2 = 5$.

SENTENCES PRESENTED ORALLY

So far we have considered children's reactions to written sentences. How do they react when these sentences are presented orally? We start again with an interview with K.

B: . . . you listen to some statements which I read, and you tell me the answer — yeah, no, true, false; whatever comes to your mind. 5 equals 2 plus 3.

K: Yeah . . . [and] . . . 'cause see, like 2 . . . 3, and here's plus and here's two [writing $3 + 2 = \square$], and then if you 1, 2, 3, and you add 2. Then, you've got 1, 2, 3, 4, 5. The answer is 5 [writes $3 + 2 = 5$].

B: Oh, I see. I'm with you. I am going to write a sentence, all right? [writes $8 = 3 + 5$] How about that?

K: But that's still not equal. That's the wrong answer. [changes $8 = 3 + 5$ to $8 + 3 = 5$ by writing + and = over the = and + signs, respectively] . . . [and] . . . you couldn't make it be 8 equals 3 plus 5 because that means that would be the answer. But it's at the wrong end!

When E is presented orally with the statement "5 is equal to 2 plus 3," she says, "It's not right" and states further, "That [3] should be a 7." It appears that she perceived "5 is equal to 2" as 5 + 2; furthermore, she did not accept the orally presented statement "3 is equal to 2 plus 1" and explained, "'Cause you put a 1 instead of a 5." On the other hand, she did accept the orally presented statement, "2 plus 1 equals 3." Again, it appears that she perceived 3 = 2 . . . as 3 + 2 . . . .

When we compare K and E, we see that E displays a rigidity of form in both written and orally presented sentences, while K displays this rigidity only in written statements.

"NON-ACTION" SENTENCES — FIRST AND SECOND GRADERS' IDEAS

The children's reactions we have considered thus far concern sentences that differ only in the order of the symbols, for example $2 + 3 = 5$ and $5 = 2 + 3$. These sentences are closer to the form most encountered by children than are such sentences as $3 = 3$, $3 = 5$, $2 + 1 = 1 + 2$, and $4 + 1 = 2 + 3$. The latter sentences either have no plus sign (e.g., $3 = 3$), or they have more than one plus sign (e.g., $2 + 1 = 1 + 2$). These sentences do not suggest an action, rather they require a judgment about their truth-value. How do children react to such sentences? We present first excerpts from an interview with K about 3 = 5 and 3 = 3.

B: What can you say about that [3 = 5]?

K: Cross that line out . . . [K writes over = to change 3 = 5 to 3 + 5].

B: Can I write this [3 = 3]? Does it make sense?

K: Nope . . . [and] . . . because, now you could fix that by going like this [changes 3 = 3 to 0 + 3 = 3] . . . 0 plus 3 equals 3.

When C is given the written statement 3 = 5, he counts on his fingers and writes 3 = 85. He is asked to read it and says, "5 . . . there's no plus, . . . [and] . . . that makes it wrong . . . [and] . . . I'll put a plus in the middle." He changes 3 = 85 to 3 + = 85 and when asked to read he says, "I'm going to read backwards, 3 plus 5 equals 8" and touches each symbol as he reads it. He responds similarly to 3 = 3: "It's wrong because there's no plus" and changes 3 = 3 to 3 + = 63. E does not accept 3 = 5 but changes it to 2 + 3 = 5 and explains, "You put a plus there and a 2 there . . . [and] . . . 'cause then that makes five." Given the written statement 3 = 3, she says, "You should put a plus here and a . . ." while she changes 3 = 3 to 0 + 3 = 3.

When T is presented with the written sentence 3 = 3, he says, "Now if you had a straight line like that [changes 3 = 3 to 3 - 3] it'd be "subtractly"; it'd be zero [proceeds to complete 3 - 3 as 3 - 3 = *0*]. When presented with 3 = 5, he responds similarly, changing 3 = 5 to 3 - 5 and then completes 3 - 5 = *0*.

These are representative reactions of first grade children. These and other interviews reveal that, when confronted with 3 = 5, they change it to 3 + 5 = 8 or 3 - 5 = 0. When presented with 3 = 3, they change it to 0 + 3 = 3, or 3 + 3 = 6, or 3 - 3 = 0. That is, each equality statement is transformed into an addition or subtraction sentence.

Do second graders display a greater maturity in this context? Let's look at responses of M and D: When M is presented with the written statement 3 = 3, he says "Yeah, 3 plus 0 does equal 3, the 0 isn't there but it's supposed to be there. And zero's not there because zero's nothing!" D explains that 3 = 3 is false because you can't add them [3 and 3]. She changes 3 = 3 to 3 + 3 = 6. Similarly, 3 = 5 is "false . . . because 3 equals 5 is not true"; she changes 3 = 5 to 3 + 5 which is "true . . . because you can add them."

Thus, both first and second graders reject sentences of the form a = b and modify them to assume the form a + b = □ or a - b = □. The latter form suggests an "action" to be performed, resulting in a sum or difference.

"NON-ACTION" SENTENCES — THIRD AND SIXTH GRADERS' IDEAS

It can be argued that the perceptions of equality sentences held by first and second graders, discussed above, are an expected outcome of the kind of instruction the children receive. But, what about third graders and above? Does the exposure to statements of commutative and associative properties broaden the children's concept of equality?

Let's examine the thinking of the third grader MA (IQ 104). When asked what 3 = 3 means, MA says, "Well, I don't know, but I can guess. It could be the end of some adding or subtracting." When asked if she would like to fix it up, she writes 3 + 0 = 3.

Does children's thinking change by the time they reach the sixth grade? TA, a sixth grader (IQ 95), when asked about the meaning of 3 = 3, says that this could mean "0 minus 3 equals 3" and writes 0 - 3 = 3. Upon further questioning, TA changes 3 = 3 to 6 - 3 = 3, then to 7 - 4 = 3, which she reads, "7 minus 4 equals 3."

The above reactions of first, second, third, and sixth graders raise some doubts about their understanding of equality. To prove the matter of understanding, it is helpful to examine symbolic representations in the context of manipulatives. Do children see the connection between statements like 3 = 3 and real objects? We present two interview episodes with K. In the final episode, K is asked to consider two piles of objects.

B: What can you say about these two piles [5 sticks and 5 beans]?

K: Like, there is 1, 2, 3, 4, 5 [counting sticks]. See they are the same thing; 3 and 2 [does ||| ||].

Now look, 1, 2, 3, and here is our two. [does ||| ooo || oo] We see that makes 5.

B: What can you say about this pile and this pile [pointing at sticks and beans]?

K: You see that pile [sticks]. If you had 3 and 2, it makes 5, you see here's my 3 [shows 3 fingers], and here's my 2 [fingers], and that makes 5 [fingers on left hand]. And here [beans] you have 3 [fingers] and 2 makes 5 [fingers on right hand].

B: What can you tell me about these two piles [5 beans; ooooo ooooo]?

K: Here's 3 and here's 3 Does [ooo 3 ||| ooo 3 |||] and then here's 2 and here's 2

Does [ooo oo 3 2 ||| || ooo oo 3 2 ||| ||] and that makes them be both be 1, 2, 3, 4, 5.

B: So what can you say about these two piles?

K: They both make 5 . . . [and] . . . you see, if I went like this . . . 1, 2, 3, 4, 5 [Does ooooo] . . . 1, 2, 3, 4, 5 [Does ooooo] . . . they're both the same thing.

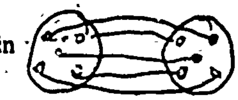
B: Can you use your pen to write something that tells me that these two piles are the same?

K: Yeah, [uses beans and traces around them to draw sets of dots: ooo oo ooo oo . . (and says)] but you see they're still in the same order. . . in the very same order.

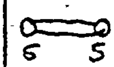
B: So then what do you need for them to be the same?

K: Like, let's say here's a thing of beans [draws ] and here's a thing of beans []. Then go like this . . . [K explains as she draws ] . .

B: Now can you write some way that tells me that there's the same number of beans here as here?

K: You see, if you went [traces lines from bean to bean in ]. You see, that matches them.

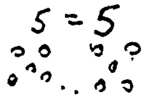
B: Does anything occur to you that you can write that tells me that these two are the same? You draw a picture. Can you write something?

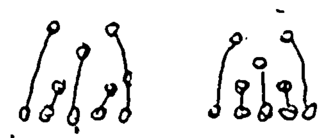
K: . . . (pause) . . . Well, you go like this writes [ 5 5].

We observe that K can demonstrate in various ways that "5 things is the same as 5 things." But she does not abstract to think 5 *equals* 5 and does not write 5 = 5. It does not appear that K has made a relationship between "is the same as" and "is equal to."

To suggest more directly an interpretation of the statements 3 = 3 and 5 = 5, the interviewer asked K to use beans.

B: Suppose that I write this [5 = 5]. Can you show me what that means with beans?

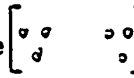
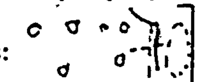
K: It means like, there's 5 beans for this number; 1, 2, 3, 4, 5 [does  5 = 5]. There's these beans does [5 = 5 ]. Then you go 1, 2, 3, 4, 5, and then you go like this 1, 2, 3, 4, 5,

[5 = 5 ]

B: If I write that [3 = 3], does that mean anything to you?

K: Yeah . . . [and] . . . you go like this [writes 3 = 3 + 0].

B: If I write that [3 = 3], can you show me what that means with beans?

K: Here's your 3 [puts beans]; then pretend like equals . . . I am not going to count that equal right now, but to pretend it's there. Then you got 3 over there []; and then you put your plus right here and then put your zero right here [writes imaginary + 0 with fingers: ] like I did there, that's the whole question.

B: [Writes $3 = 3$] O. K. Can you show me what that means with beans?

K: That means that you have 3 beans over here [does $3 = 3$] and that means they are equal in number.

We observe that K is able to match a numeral with a set of objects and a set of objects with a numeral. However, it does not appear that K is able to abstract the relational statement $3 = 3$ from her own demonstrations of the sameness of two sets of three. Thus, it is not clear that K is able to "move" back and forth between object demonstrations and corresponding relational statements. However, we can observe from K's reaction to manipulatives a degree of potential for adequate understanding that is not evident from her reaction to written mathematical statements. It does appear that K has a good internalized concept of sameness as it relates to sets of objects. What K appears to need is a teacher-manipulated learning environment which will provide her with situations which force a need to translate the demonstrations into symbolic sentences.

A FURTHER LOOK AT EQUALITY

We remarked earlier that children may react in various ways to written and orally given sentences. A first grader C was presented orally with *3 equals 5* and asked to write it *as it is read to you.* He wrote $3 + 5 =$. When the interviewer stressed 3 . . . equal . . . 5, C wrote: $= 3$ and then $= 5\ 3$; and then said, "Oops, you forgot the plus" and wrote: $= 5 + 3$. When the interviewer read very slowly to C, 3 equals 3, C wrote $3 +$, stopped and said, "I messed up too. You said equal." He wrote $3 = 3$, read "3 equal 3" and suddenly, "fault, that's a fault . . . 'cause that [=] ain't supposed to be in the middle."

A sixth grader D (IQ 131) had no difficulty accepting and writing equality statements of the form discussed above. However, when encountering fractions, he revealed his concept of equality to be in contrast to the concept of *sameness*. In this interview excerpt, N is the interviewer.

N: What about if I have a number like that [writes $\frac{6}{2}$]; what's that?

D: An improper fraction.

N: Is that a whole number?

D: Unless you change it—you can change it into a whole number.

N: What whole number is that?

D: Three.

N: Three. If I'd ask you: six-halves, is that a whole number, what would you say?

D: No.

N: Now, let's see [writes $\frac{6}{2} = 3$]. Is that O. K.? Is that right?

D: Yeah.

N: Six halves equals three. This is a whole number [points to 3 in $\frac{6}{2} = 3$], but six halves is not [points to $\frac{6}{2}$ in $\frac{6}{2} = 3$].

D: [Nods his head in agreement.]

N: Now that kind of confuses me, to tell you the truth, because if this is a whole number [points to 3 in $\frac{6}{2} = 3$] and I think you told me when I wrote something like this [writes 3 = 3] you said this and this [points to each 3 in 3 = 3] are the same number. Now doesn't this [points to $\frac{6}{2}$ and 3 in $\frac{6}{2} = 3$] say that this is the same number?

D: Uh-uh. It just says that they are equal to each other, they have the same value, but that doesn't mean that they are the same number.

N: So, that [points to $\frac{6}{2} = 3$] doesn't mean that they are the same number.

D: Uh-huh.

We see that D distinguishes between the concept of equality (the same value) and the concept of sameness: to be equal does not mean to be the same. Thus, to D, $\frac{6}{2}$ and 3 are two different numbers, having the same value. Furthermore, 3 is a whole number and $\frac{6}{2}$ is not; it is an improper fraction. Later in the interview he asserts that no fraction is a whole number.

EQUALITY STATEMENTS WITH TWO PLUS SIGNS

We next examine children's ideas about such statements as 2 + 3 = 3 + 2. These are relational statements which differ from such statements as 3 + 2 = 5 in that they have two plus signs. Again we begin with an interview episode with K.

K accepts 2 + 3 = 5 and 3 + 2 = 5 saying:

K: Yes, 'cause see, they're the same question [pointing at the 3's and 2's in pairs].

B: Oh, I see. All right, now what about this [writes 2 + 3 = 3 + 2].

K: . . . (pause) . . . nope . . . [and] . . . 'cause see, you wrote two of the same thing. You didn't . . . you didn't see, you put 2 plus 3 equals 3 plus 2 . . . [then] . . . you didn't do it right [K changes 2 + 3 = 3 + 2 to 2 + 3 = 5 and 3 + 2 = 5].

B: . . . Can I write [writes 1 + 5 = 5 + 1]?

K: No . . . [and] . . . this should be an answer right here [does 1 + 5 = 1 + 5]; and right there should be the equal [does 1 + 5 = 1 + 5 =] and the answer.

Two other first graders, C and E, reverted to statements of the same form as follows: When C is presented with the written sentence 1 + 2 = 2 + 1, he changes this to $\begin{matrix}1+2=2+1\\ 3\quad\;\; 3\end{matrix}$. He reads $\begin{matrix}1+2=2+1\\ 3\quad\;\; 3\end{matrix}$ as follows: "1 plus 2 equals 3" [touching the symbols in $\begin{matrix}1+2=\\ 3\end{matrix}$ as they are read] and "1 plus 2 equals 3" [touching the symbols in $\begin{matrix}=2+1\\ 3\end{matrix}$ as each is read]. First grader E accepts 2 + 3 = 5 and 3 + 2 = 5, but not 3 + 2 = 2 + 3 "because it [2 + 3] doesn't have anything . . . [and] . . . 'cause . . . [you] forgot to put the 5." She writes 3 + 2 = 5 and 2 + 3 = 5.

Here are some of the reactions of second graders. M accepts 3 + 2 = 2 + 3 "because they both have the same numbers. Only 2 plus 3 is backwards." M sees a difference between 3 + 2 = 2 + 3 and 4 + 1 = 2 + 3 explaining that "Two four [pointing at 2 and 4] one three [pointing at 1 and 3] doesn't rhyme. There should be a 3 here [touching 4] and a 2 there [touching 1], or a 2 here [touching 4] and a 3 here [touching 1]. They are sort of equal because they both equal, both equal 5. They don't go together; not made the same . . . [then] . . . they both equal 5, but they're not the same." D does not accept 1 + 4 = 4 + 1 "because you need a plus there [=]." Similarly, she does not accept 3 + 2 = 2 + 3 "because you need to change it [=] to plus." She changes 3 + 2 = 2 + 3 to 3 + 2 + 2 + 3 = 10.

A third grader M is presented with 3 + 2 = 2 + 3 and asked to read it. She reads, "3 plus 2 equal," and after a pause, "No, 3 plus 2 equals," again a long pause, "2 plus 3." When asked if that means anything, she says, "I don't know." When asked if she could tell what this means to a younger brother or sister, she holds her hand over [2 + 3 in 3 + 2 = 2 + 3] and says, "You could add these" [tracing over 3 + 2 =] and then covers 3 + 2 = in 3 + 2 = 2 + 3 and makes the same observation about 2 + 3.

The essential observation to be made here is that these children do not view sentences like 2 + 3 = 3 + 2 as being sentences about number relationships. They do not see these as indicating the sameness of sets of objects. Indeed, it appears that the children considered these as "do something" sentences. In most cases the presence of a plus sign along with two numerals suggests that another number is to be found. Moreover, in at least the case of M, it does appear that he is concerned about the "sameness" of two expressions in a symbolic sense. To him 2 + 3 = 3 + 2 is O. K., because the two expressions 2 + 3 and 3 + 2 have the same numerals, but 4 + 1 = 2 + 3 is not O. K., because they don't have the same numerals—"They don't rhyme."

Finally, let's observe K's responses to sentences like 5 + 2 = 3 + 4 and how they differ with the mode of presentation, oral or in writing. We present a portion of an interview.

B: Say we have 5 plus 2 . . . [writes 5 + 2 = $\boxed{}$]. I'm going to finish it like this [writes 5 + 2 = $\boxed{3 + 4}$].

K: . . . (pause) . . . right here [does 5 + 2 = $\boxed{3 + 4}$] should be your answer, in the box (☐); and then this problem [touches 3 + 4] should be down here [writes 3 + 4 below] . . . [and] . . . now, when you get this box [i.e., 5 + 2 = ☐], go like this [writes 3 + 4 = ☐]. Then you put your answer in here [i.e., in the ☐'s]. You don't write another problem [i.e., after the = sign].

B: So I understand when you see something like 5 + 2 = 3 + 4 [writing 5 + 2 = 3 + 4] that you really think of that as two problems?

K: Yeah . . . [and] . . . this problem [5 + 2] makes 7 and this problem [3 + 4] makes 7. [K writes
5 + 2 = 3 + 4]
7 7

Again, as we observed earlier, K has definite ideas about how sentences should be written.

Now let's observe how K reacts to similar statements given orally.

B: . . . Now let me ask you again, O. K.? Four plus five equals three plus six?

K: Yeah.

B: O. K. Now, when I write [referring to an earlier response] 4 plus 5 equals 3 plus 6 [writing 4 + 5 = 3 + 6], you say no. How come when I read it you say yeah; and when I write it you say no? I'm curious.

K: 'Cause that [points after = in 4 + 5 = 3 + 6] should be your answer. It's the end, not another problem. Like if you went into writing you'd go like this [writes 4 + 5 = 9; 3 + 6 = 9].

K accepts orally presented statements such as *5 plus 2 equals 3 plus 4* more readily than she does the corresponding statements given in written form. A clue as to why she behaves this way is found in the explanation of what she would do with *4 plus 5 equals 3 plus 6* "if you went to writing": She transforms $4 + 5 = 3 + 6$ into two statements, $4 + 5 = 9$ and $3 + 6 = 9$; then she compares the results of those statements.

SUMMARY AND IMPLICATIONS

The question which originally motivated this clinical interview study about children's understanding of various forms of equality sentences was whether children consider equality to be an operator or a relation. As an operator symbol, = would be a "do something signal." As a relational symbol, = suggests a comparison of the two members of an equality sentence. The interview data presented suggest that children consider the symbol = as a "do something signal." There is a strong tendency among all of the children to view the = symbol as being acceptable in a sentence only when one (or more) operation signs (+, ·, etc.) precede it. Some children, in fact, tell us that the answer must come after the =. We observe in the children's behavior an extreme rigidity about written sentences, an insistence that statements be written in a particular form, and a tendency to perform actions (e.g., add) rather than to reflect, make judgments, and infer meanings. Moreover, it is important to observe that there is no evidence to suggest that children change in their thinking about equality as they get older and progress to upper grades; in fact, the evidence seems to be to the contrary.

What are the implications of this information to the teachers of mathematics? Is the conception of equality exhibited by these children satisfactory or does it suggest deep-seated notions which may (a) cause difficulty in learning other mathematical concepts which involve equations, or (b) be related to other, and possibly even more basic concepts? It is possible that the limited and rigid concept of inequality that these children exhibit leads to misunderstanding when children deal with problems like $8 + 7 = 8 + (\_\_\_\_ + \_\_\_\_)$, etc. which in many current curricular materials appear as a prerequisite concept for place value work. It is also possible that this limited concept of equality interferes with the later development of algorithms. Consider, for example, what such a sequence as

$$\begin{array}{r@{\;}l} 47 & = 40 + 7 \\ \underline{+32} & = \underline{30 + 2} \\ & = 70 + 9 \\ & = 79 \end{array}$$

means to a child whose concept of equality is essentially that of an operator symbol.

It is also possible, that the children's concept of equality as an operator symbol, rather than a relational symbol, is symptomatic of their limited understanding and experience with relational terms in general, such as *same, more, less, as many as*, etc. This brings one very close to developmental, linguistic, and other considerations which go beyond the intent of this paper.

The authors take the point of view that the behavior displayed by these children represents a very incomplete conception—indeed a misconception from a mathematical point of view—about a basic concept of mathematics. Thus, the conclusion that we must attend to the problem of teaching children a more adequate concept of equality, and possibly relational concepts in general, is inescapable. One must be careful not to conclude that simply making sure that children are exposed to the various forms of equality sentences will remove the problem. The behaviors uncovered in this investigation suggest a deep-seated mind set which produces rigid reactions, particularly to written number sentences. Inherent in this is a more general question about how children attach meanings to symbols and how they acquire relational concepts.